ME617 - Handout 14

Vibrations of Continuous Systems

Axial vibrations of elastic bars

The figure shows a uniform elastic bar of length L and cross section A. The bar material properties are its density ρ and elastic modulus E. One end of the bar is attached to a fixed wall while the other end is free. The force $P(t)$ acting at the free end of the bar induces elastic displacements $u(x,t)$ along the bar

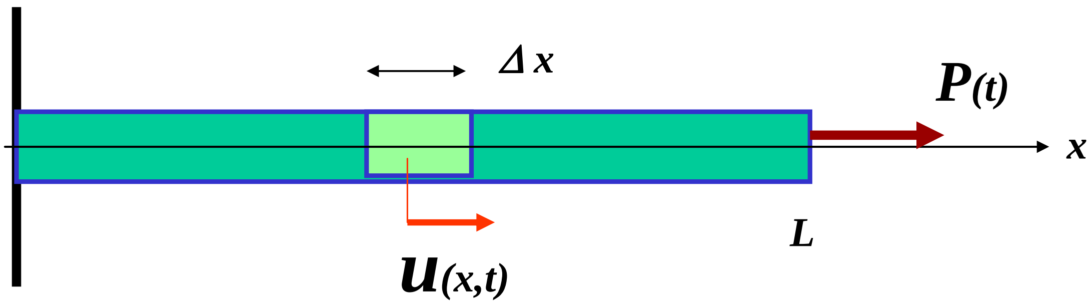


Fig. Schematic view of elastic bar undergoing axial motions

From elementary strength of materials consider

a) Cross-sections A remain plane and perpendicular to the main axis (x) of the bar.

b) Material is linearly elastic

c) Material properties (ρ, E) are constant at any given cross section.

The relationship between stress σ and strain ε for uniaxial tension is

$$\sigma = E\,\varepsilon = E\frac{\partial u}{\partial x} \qquad (1)$$

Consider the free body diagram of an infinitesimally small piece of bar with length Δx,

In the FBD, $P(x,t) = A_{(x)}\,\sigma = A\,E\,\frac{\partial u}{\partial x}$ is the axial force at a cross section of the bar, and $f(x,t)$ is a distributed axial force per unit length,

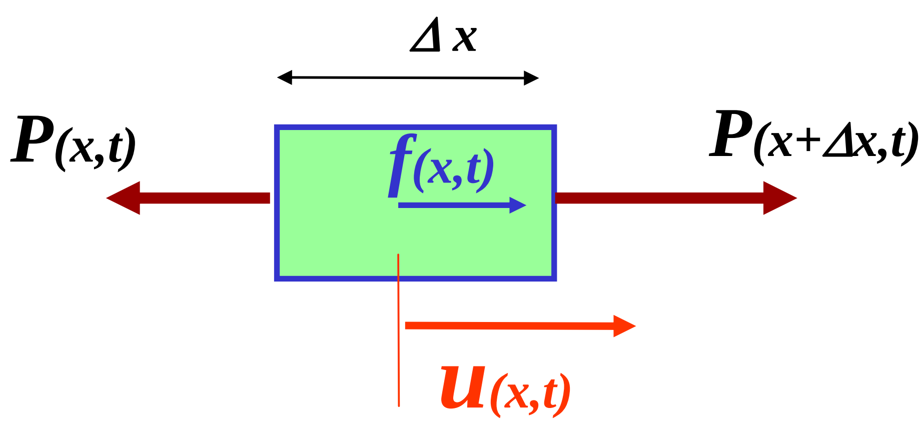


Fig. Free body diagram of small piece of elastic bar

Applying Newton's 2nd law of motion on the bar differential element gives

$$\sum_x F_x = \Delta m\, a_x = (\rho\, A\, \Delta x)\frac{\partial^2 u}{\partial t^2} \tag{2}$$

$$(\rho\, A\, \Delta x)\frac{\partial^2 u}{\partial t^2} = P_{(x+\Delta x,t)} - P_{(x,t)} + f_{(x,t)}\Delta x \tag{3}$$

As $\Delta x \to 0 \;\; \Rightarrow P_{(x+\Delta x,t)} \approx P_{(x,t)} + \frac{\partial P}{\partial x}\Delta x$ (4a)

$$\left(\rho A \Delta x\right)\frac{\partial^2 u}{\partial t^2}=\frac{\partial P}{\partial x}\Delta x + f_{(x,t)}\Delta x$$

$$\rho A \frac{\partial^2 u}{\partial t^2}=\frac{\partial P}{\partial x} + f_{(x,t)} \tag{4}$$

And replacing $P(x,t)=AE\frac{\partial u}{\partial x}$

$$\rho A \frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial x}\left(AE\frac{\partial u}{\partial x}\right) + f_{(x,t)} \tag{5}$$

PDE (5) describes the axial motions of an elastic bar. For its solution, one needs appropriate boundary conditions *(BC)*, which are of two types

(a) **essential**, $u=u_*$, a specified value, at $x=x_*$ for all times,

(b) **natural,** $P(x_*,t) = \left. AE\frac{\partial u}{\partial x}\right)_{x=x_*}$ specified

If ***P=0***, then the natural BC is a **free end**, i.e. $\left.\frac{\partial u}{\partial x}\right)_{x=x_*}=0$

Note: PDE (5) and its BCs can be derived from the Hamiltonian principle using the definitions for kinetic (*T)* and potential (*V)* energies.

$$T=\frac{1}{2}\int_0^L \rho A\left(\frac{\partial u}{\partial t}\right)^2 dx; \quad V=\frac{1}{2}\int_0^L EA\left(\frac{\partial u}{\partial x}\right)^2 dx \tag{6}$$

Free vibrations of elastic bars

Without external forces (point loads or distributed load, *f*=0), PDE (5) reduces to

$$\rho A \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(A E \frac{\partial u}{\partial x} \right) \tag{7}$$

The solution of PDE (7) is of the form $u_{(x,t)} = \phi_{(x)} v_{(t)}$ (8)

Note that

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= \phi_{(x)} \frac{d^2 v}{d t^2} = \phi_{(x)} \ddot{v}_{(t)} \; ; \\ \frac{\partial^2 u}{\partial x^2} &= \frac{d^2 \phi}{d x^2} v_{(t)} = \phi'' v_{(t)} \end{aligned} \tag{9}$$

With the definitions $\left(\dot{} \right) = d/_{dt}$; $\left(' \right) = d/_{dx}$. For a bar with uniform material properties (ρ, E) and cross section A, substitution of the product solution Eq. (8) into PDE (7) gives

$$\frac{\rho}{E} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \rightarrow \quad \frac{\rho}{E} \phi_{(x)} \ddot{v}_{(t)} = \phi''_{(x)} v_{(t)} \tag{10}$$

Divide this expression by $u_{(x,t)} = \phi_{(x)} v_{(t)}$ to get

$$\frac{\ddot{v}_{(t)}}{v_{(t)}} = \frac{E}{\rho} \frac{\phi''_{(x)}}{\phi_{(x)}} \tag{11}$$

Above, the *LHS* is only a function of time, while the *RHS* is only a function of spatial coordinate *x*. This is possible only if both sides equal to a constant, i.e.

$$\frac{\ddot{v}_{(t)}}{v_{(t)}} = \frac{E}{\rho}\frac{\phi''_{(x)}}{\phi_{(x)}} = -\omega^2$$

Hence, the PDE is converted into two ordinary differential equations (ODEs), i.e.

$$\begin{aligned} \ddot{v}_{(t)} + \omega^2 v &= 0 \\ \phi''_{(x)} + \lambda^2 \phi_{(x)} &= 0 \end{aligned} \tag{12}$$

where
$$\lambda = \omega\sqrt{\rho/E} \tag{13}$$

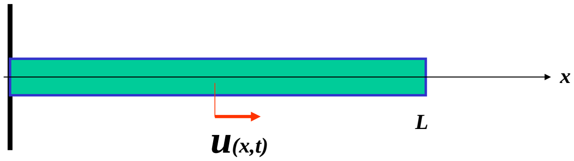
The solution of the ODEs (12) & (13) is

$$v_{(t)} = C_t \cos(\omega t) + S_t \sin(\omega t) \tag{14}$$

$$\phi_{(x)} = C_x \cos(\lambda x) + S_x \sin(\lambda x) \tag{15}$$

The coefficients (*C, S*) are determined from satisfying the boundary conditions for the specific bar configuration and load condition. Equation (15) is known as the **fundamental equation** for an elastic bar, i.e. it contains the information on natural frequencies and mode shapes.

Example 1.



A bar with one end fixed and the other end free.
In this case, the boundary conditions are

At $x=0$, $$u_{(0,t)}=0=\phi_{(0)}v_{(t)} \Rightarrow \phi_{(0)}=0 \quad \forall t$$

At $x=L$, $$\left.\frac{\partial u}{\partial x}\right]_{x=L}=0=\phi'_{(L)}v_{(t)} \Rightarrow \phi'_{(L)}=0 \quad \forall t \qquad (16)$$

Hence, from the characteristic equation $\phi_{(0)}=0 \to C_x=0$ and

$$\phi_{(x)}=S_x \sin(\lambda x) \qquad (17)$$

At $x=L$, $$\phi'_{(L)}=0=\lambda S_x \cos(\lambda L)=0 \qquad (18)$$

Note that $S_x \neq 0$ for a non trivial solution. Hence, the **characteristic equation** for axial motions of a **fixed end-free end elastic bar** is

$$\cos(\lambda L)=0 \qquad (19)$$

which has an infinite number of solutions, i.e.

$$\lambda L=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\ldots.,\infty=\frac{2n-1}{2}\pi, \; {}_{n=1,2,\ldots..}$$

And hence the roots of Eq. (19) are

$$\lambda_n=\frac{(2n-1)}{2}\frac{\pi}{L} \qquad {}_{n=1,2,\ldots..} \qquad (20)$$

And since $\lambda = \omega\sqrt{\rho/E}$, the natural frequencies of the fixed end-free end bar are

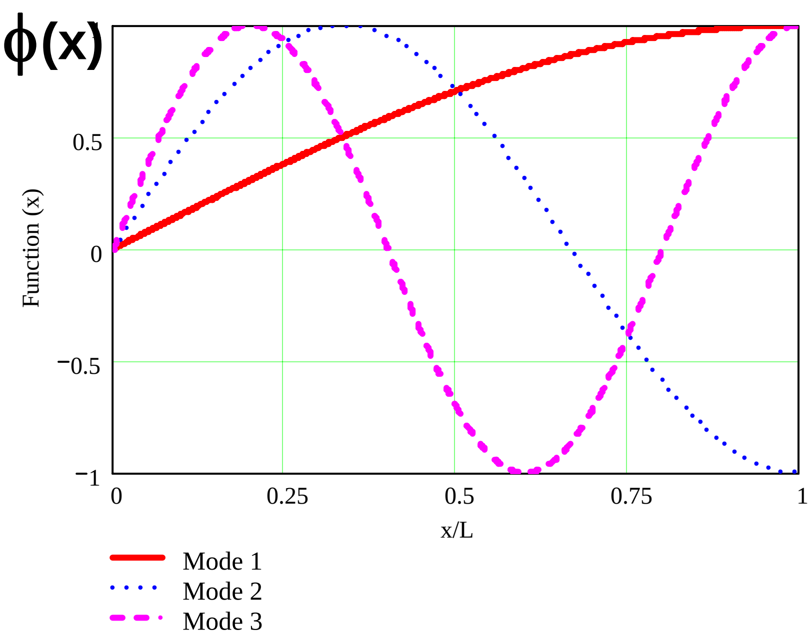
$$\omega_k = \frac{(2k-1)}{2}\frac{\pi}{L}\left(\frac{E}{\rho}\right)^{1/2} \quad ; \quad _{k=1,2,.....} \tag{21}$$

i.e. $\omega_1 = \frac{\pi}{2L}\left(\frac{E}{\rho}\right)^{1/2}$, $\omega_2 = \frac{3\pi}{2L}\left(\frac{E}{\rho}\right)^{1/2}$, $\omega_3 = \frac{5\pi}{2L}\left(\frac{E}{\rho}\right)^{1/2}$ ....

Associated to each natural frequency, there is a **natural mode shape**

$$\phi_k = \psi_k = \sin(\lambda_k\, x) \quad _{k=1,2,.....} \tag{22}$$

as shown in the figure below.



$$\lambda = \begin{pmatrix} 1.571 \\ 4.712 \\ 7.854 \\ 10.996 \end{pmatrix} \frac{1}{L}$$

Fig. Natural modes shapes ϕ(x) for elastic bar with fixed end-free end

See more examples on page 13-ff.

The **displacement function response** $u_{(x,t)} = \phi_{(x)} v_{(t)}$ equals to the superposition of all the found responses, i.e.

$$u_{(x,t)} = \sum_k \phi(x)_k \, v(t)_k =$$

$$\sum_{k=1}^{\infty} \phi_{(x)_k} \left[C_k \cos(\omega_k t) + S_k \sin(\omega_k t) \right] \tag{23a}$$

For example 1 (fixed end –free end bar)

$$u_{(x,t)} = \sum_{k=1}^{\infty} \sin(\lambda_k x) \left[C_k \cos(\omega_k t) + S_k \sin(\omega_k t) \right] \tag{23b}$$

and velocity:

$$\dot{u}_{(x,t)} = \sum_{k=1}^{\infty} \sin(\lambda_k x) \omega_k \left[-C_k \sin(\omega_k t) + S_k \cos(\omega_k t) \right] \tag{24}$$

The set of coefficients (C_k, S_k) are determined by satisfying the initial conditions. That is at time $t=0$,

$$u_{(x,0)} = U_{(x)} = \sum_{k=1}^{\infty} \sin(\lambda_k x) C_k$$

$$\dot{u}_{(x,0)} = \dot{U}_{(x)} = \sum_{k=1}^{\infty} \omega_k \sin(\lambda_k x) S_k \tag{25}$$

Orthogonality properties of the natural modes

Recall that the pair $\left\{\lambda_k, \psi_{(x)_k}\right\}_{k=1,\ldots\infty}$ satisfy the characteristic equation (12b), i.e.

$$\psi''_{(x)_k} + \lambda_k^2 \psi_{(x)_k} = 0 \qquad k=1,2\ldots\infty \tag{26}$$

And consider two **different** eigenvalues λ_i and λ_j each satisfying Eq. (26), i.e.

$$\psi''_i + \lambda_i^2 \psi_i = 0 \quad \& \quad \psi''_j + \lambda_j^2 \psi_j = 0$$

Multiply Eq. on left by ψ_j and Eq. on right by ψ_i, and integrate over the domain $x \in \{0, L\}$ to get:

$$\int_0^L \left(\psi_j \psi''_i \, dx\right) + \lambda_i^2 \int_0^L \left(\psi_j \psi_i \, dx\right) = 0$$

$$\int_0^L \left(\psi_i \psi''_j \, dx\right) + \lambda_j^2 \int_0^L \left(\psi_i \psi_j \, dx\right) = 0 \tag{27}$$

Integrate by parts the term on the LHS to obtain

$$\int_0^L \psi_j \psi''_i \, dx = \left(\psi_j \psi'_i\right]_{x=0}^{x=L} - \int_0^L \psi'_j \psi'_i \, dx \tag{28}$$

And recall the boundary conditions for the **fixed end-free end bar**

$$\left(\psi_j\right]_{x=0} = 0 \quad \& \quad \left(\psi'_i\right]_{x=L} = 0 \tag{29}$$

And write first of Eq. (27) as $\lambda_i^2 \int_0^L \left(\psi_j \psi_i \, dx\right) = \int_0^L \left(\psi'_j \psi'_i \, dx\right)$ and substituting $\lambda_i = \omega_i \sqrt{\rho / E}$ one obtains:

$$\omega_i^2 \int_0^L \left(\rho A \psi_j \psi_i \, dx\right) = \int_0^L \left(E A \psi'_j \psi'_i \, dx\right) \tag{30a}$$

$$\omega_j^2 \int_0^L \left(\rho A \psi_i \psi_j \, dx\right) = \int_0^L \left(E A \psi'_i \psi'_j \, dx\right) \tag{30b}$$

Subtract Eq. (30b) from (30a) to obtain

$$\left(\omega_j^2 - \omega_i^2\right) \int_0^L \left(\rho A \psi_i \psi_j \right) dx = 0 \tag{31}$$

And since $\omega_i \neq \omega_j$, it follows that

$$\int_0^L \left(\rho A \psi_i \psi_j \right) dx = 0 \;\; \& \;\; \int_0^L \left(E A \psi'_i \psi'_j \right) dx = 0 \quad {}_{i \neq j = 1,2,....\infty} \tag{32}$$

That is, the modal functions $\left\{\psi_k\right\}_{k=1,2...}$ are **ORTHOGONAL.** For *i=j,* the i_{th} natural frequency follows from

$$\omega_i^2 = \frac{\text{K}_i}{\text{M}_i} = \frac{\int_0^L \left(E A \psi'_i \psi'_i \right) dx}{\int_0^L \left(\rho A \psi_i \psi_i \right) dx} \tag{33}$$

Where K_i, M_i are the i_{th} mode *equivalent* stiffness and mass coefficients.

Note that the set $\{\psi_k\}_{k=1,2...}$ is a **COMPLETE SET** of orthogonal functions

Now, consider the initial conditions, Eq. (25)

$$u_{(x,0)} = U_{(x)} = \sum_{k=1}^{\infty} \sin(\lambda_k x) C_k$$

$$\dot{u}_{(x,0)} = \dot{U}_{(x)} = \sum_{k=1}^{\infty} \omega_k \sin(\lambda_k x) S_k \qquad (25)$$

Multiply both sides of Eq. (25) by $\psi_m = \sin(\lambda_m x) \times \rho A$ and integrate over the whole domain to obtain

$$\int_0^L \left(\rho A \psi_m U_{(x)}\right) dx = \sum_{k=1}^{\infty} C_k \int_0^L \left(\rho A \psi_m \psi_k\right) dx$$

And since

$$\int_0^L \left(\rho A \psi_m \psi_k\right) dx = \begin{Bmatrix} \mathrm{M}_m & \text{when } m=k \\ 0 & \text{when } m \neq k \end{Bmatrix} \qquad (34)$$

Then if follow that

$$C_m = \frac{\int_0^L \left(\rho A \psi_m U_{(x)}\right) dx}{\mathrm{M}_m}, \quad m=1,2,...\infty \qquad (35)$$

And similarly

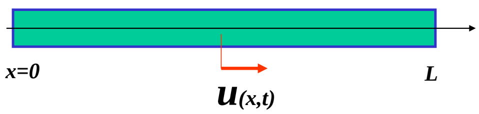
$$S_m = \frac{\int_0^L \left(\rho A \psi_m \dot{U}_{(x)}\right) dx}{\omega_m \mathrm{M}_m}, \quad m=1,2,...\infty \qquad (36)$$

with $M_m = \int_0^L \left(\rho A \psi_m^2\right) dx$ and $K_m = \int_0^L \left(E A \left[{}^{d\psi_m}\!/\!_{dx}\right]^2\right) dx$ (37)

This concludes the procedure to obtain the full solution for the vibrations of a bar, i.e.

$$u_{(x,t)} = \sum_{k=1}^{\infty} \phi_{(x)_k} \left[C_k \cos(\omega_k t) + S_k \sin\left(\omega_k t\right) \right] \quad (23)$$

Example 2.



A bar with both ends free.
The boundary conditions are

At $x=0$, $$\left.\frac{\partial u}{\partial x}\right]_{x=0} = 0 = \phi'_{(0)} v_{(t)} \Rightarrow \quad \phi'_{(0)} = 0 \quad \forall t$$

At $x=L$, $$\left.\frac{\partial u}{\partial x}\right]_{x=L} = 0 = \phi'_{(L)} v_{(t)} \Rightarrow \quad \phi'_{(L)} = 0 \quad \forall t$$

Hence, from the characteristic equation $\phi'_{(0)} = 0 \rightarrow S_x = 0$ and

$$\phi_{(x)} = C_x \cos(\lambda x)$$

At $x=L$, $$\phi'_{(L)} = 0 = \lambda C_x \sin(\lambda L) = 0$$

Note that $\lambda = 0$ denotes rigid body motion. Hence, the **characteristic equation** for axial motions of a **fixed end-free end elastic bar** is

$$\sin(\lambda L) = 0$$

which has an infinite number of solutions, i.e.

$$\lambda L = 0, \pi, 2\pi, 3\pi, ..., \infty = n\pi, \;_{n=0,1,2.....}$$

$$\lambda_n = n\frac{\pi}{L} \qquad _{n=0,1,2,.....}$$

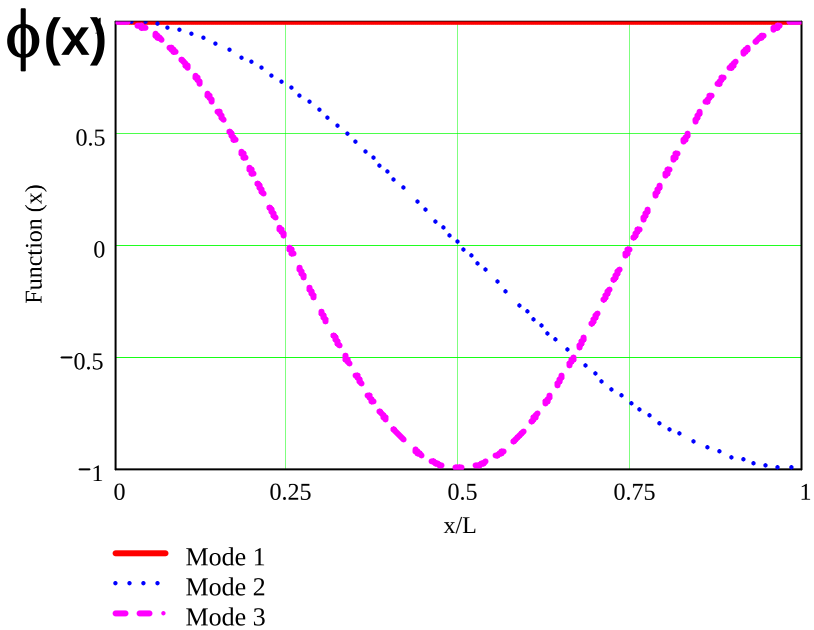
And since $\lambda = \omega\sqrt{\rho/E}$, the natural frequencies of the free end-free end bar are

$$\omega_k = k\frac{\pi}{L}\left(\frac{E}{\rho}\right)^{1/2} \quad ; \quad {}_{k=0,1,2,.....}$$

Associated to each natural frequency, there is a **natural mode shape**

$$\phi_k = \cos\left(\lambda_k\, x\right) \quad {}_{k=0,1,2,...}$$

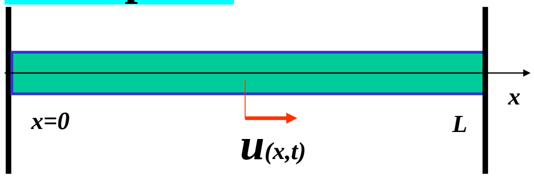
And shown in the figure below.



$$\lambda = \begin{pmatrix} 0 \\ 3.142 \\ 6.283 \end{pmatrix} \frac{1}{L}$$

Fig. Natural modes shapes ϕ(x) for elastic bar with both ends free. First mode is rigid body (null natural frequency)

Example 3.



A bar with both ends fixed.
The boundary conditions are

At $x=0$, $\quad u_{(0,t)} = 0 = \phi_{(0)} v_{(t)} \Rightarrow \phi_{(0)} = 0 \quad \forall t$

At $x=L$, $\quad u_{(L,t)} = 0 = \phi_{(L)} v_{(t)} \Rightarrow \phi_{(L)} = 0$

Hence, from the characteristic equation $\phi_{(x)} = C_x \cos(\lambda x) + S_x \sin(\lambda x)$, then $\phi_{(0)} = 0 \rightarrow C_x = 0$ and

$$\phi_{(x)} = S_x \sin(\lambda x)$$

At $x=L$, $\qquad \phi_{(L)} = 0 = \sin(\lambda L) = 0$

Note that $\lambda \neq 0$ denotes rigid body motion. Hence, the **characteristic equation** for axial motions of a **fixed end-fixed end elastic bar** is

$$\sin(\lambda L) = 0$$

which has an infinite number of solutions, i.e.

$$\lambda L = \pi, 2\pi, 3\pi, ..., \infty = n\pi, \quad _{n=0,1,2.....}$$

$$\lambda_n = n\frac{\pi}{L} \qquad _{n=1,2,.....}$$

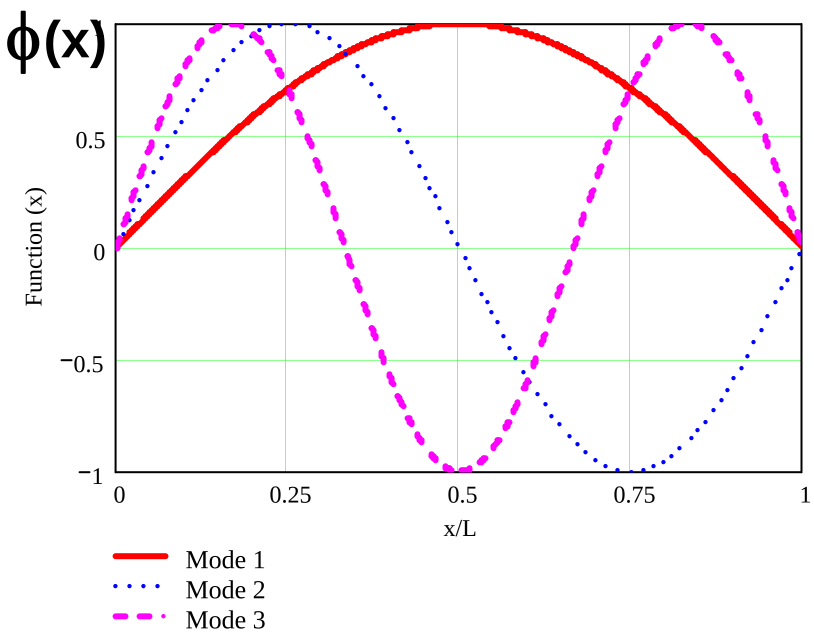
And since $\lambda = \omega \sqrt{\rho/E}$, the natural frequencies of the free end-free end bar are

$$\omega_k = k\frac{\pi}{L}\left(\frac{E}{\rho}\right)^{1/2} \quad ; \quad _{k=1,2,.....}$$

Associated to each natural frequency, there is a **natural mode shape**

$$\phi_k = \sin\left(\lambda_k\, x\right) \quad {}_{k=0,1,2,\ldots}$$

And shown in the figure below.



$$\lambda = \begin{pmatrix} 3.142 \\ 6.283 \\ 9.425 \end{pmatrix} \frac{1}{L}$$

Fig. Natural modes shapes ϕ(x) for elastic bar with both ends fixed.

Vibrations of Continuous Systems

Lateral vibrations of elastic beams

The figure shows a uniform elastic beam of length *L*, cross section *A* and area moment of inertia *I*. The beam material properties are its density ρ and elastic modulus *E*. One end of the beam is fixed to a wall while the other end is free. The discrete force *P(t)* acts at a fixed axial location while *f(x,t)* represents a load distribution per unit length. The forces induces elastic displacements on the beam and designated as *v(x,t)*.

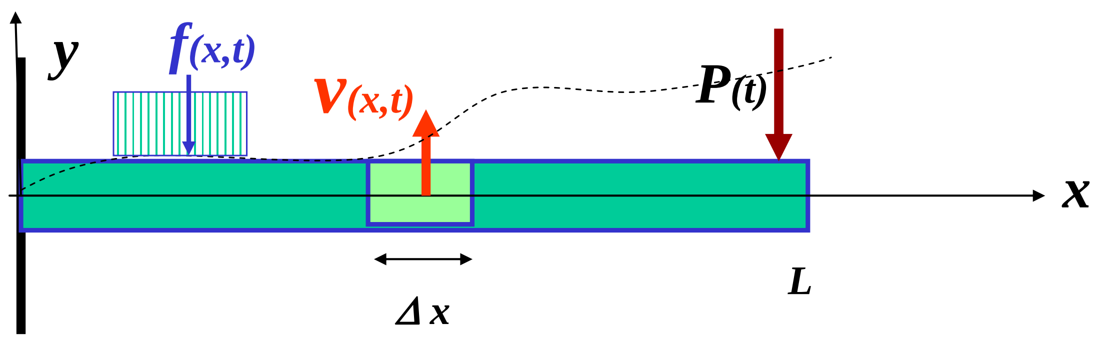


Fig. Schematic view of elastic beam undergoing lateral motions

From elementary strength of materials consider

a) Cross-sections *A* remain plane and perpendicular to the neutral axis (*x*) of the beam.

b) Homogeneous material beam, linearly elastic,

c) Material properties (ρ, E) are constant at any given cross section.

d) Stresses σ_y , σ_z << σ_x (flexural stress), i.e. along beam.

The graph below shows the free body diagram for motion of a differential beam element with length Δx.

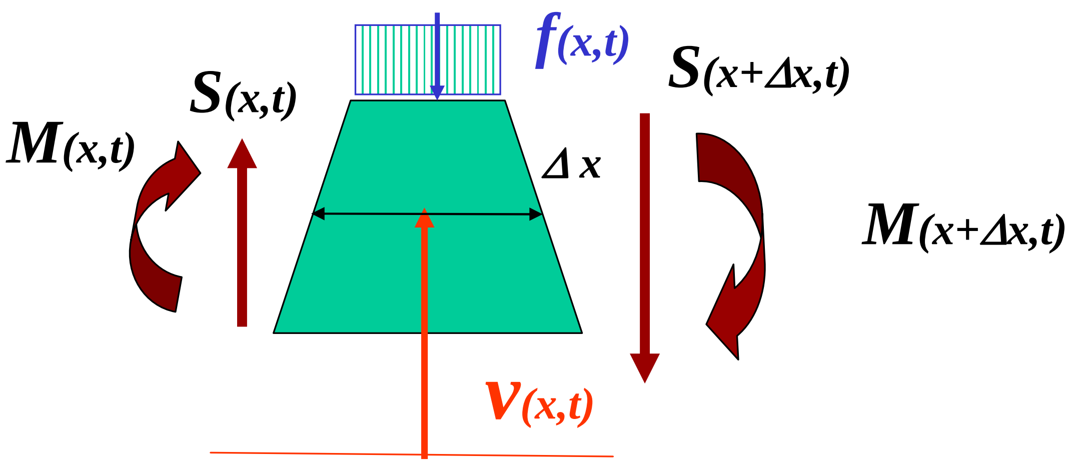


Fig. Free body diagram of small piece of elastic beam

In The *FBD*, $S_{(x,t)}$ represents the shear force and $M_{(x,t)}$ denotes the bending moment. Apply Newton's 2nd law to the material element:

$$\sum_x F_x = \Delta m\, a_x = S - \left(S + \frac{\partial S}{\partial x} \Delta x \right) + f_{(x,t)} = (\rho A \Delta x) \frac{\partial^2 v}{\partial t^2} \quad (38)$$

In the limit as $\Delta x \to 0$:

$$(\rho A) \frac{\partial^2 v}{\partial t^2} = f_{(x,t)} - \frac{\partial S}{\partial x} \quad (39)$$

Apply the moment equation: $\sum M = \Delta I_g \ddot{\alpha} \sim 0$ (40)

neglecting rotary inertia ΔI_g

Then $$\sum M \approx 0 = M_{(x+\Delta x,t)} - M_{(x,t)} - f\frac{\Delta x^2}{2} - S\,\Delta x$$
$$= M + \frac{\partial M}{\partial x} - M - f\frac{\Delta x^2}{2} - S\,\Delta x$$

In the limit as $\Delta x \to 0$: $$\boxed{\frac{\partial M}{\partial x} = S_{(x,t)}} \quad (41)$$

Combining Eqs. (41) and (39) gives:

$$\boxed{(\rho A)\frac{\partial^2 v}{\partial t^2} = f_{(x,t)} - \frac{\partial^2 M}{\partial x^2}} \quad (42)$$

If the slope $\left({}^{\partial v}\!/_{\partial x}\right)$ remains small, then the beam curvature is ${}^{1}\!/_{\tilde{\rho}} = {}^{\partial^2 v}\!/_{\partial x^2}$. From **Euler's beam theory**:

$$\boxed{M = \frac{E\,I}{\tilde{\rho}} = E\,I\frac{\partial^2 v}{\partial x^2}} \quad (43)$$

where $I = \oiint \rho\, y^2\, dA$ is the beam area moment of inertia. Substitute Eq. (43) into (42) to obtain the **equation for lateral motions of an elastic beam**:

$$\boxed{(\rho A)\frac{\partial^2 v}{\partial t^2} = f_{(x,t)} - \frac{\partial^2}{\partial x^2}\left(E\,I\frac{\partial^2 v}{\partial x^2}\right)} \quad (44)$$

The PDE is fourth-order in space and 2nd order in time. Appropriate boundary conditions are of two types:

Essential BCs:

- specified displacement, $v = v_*$

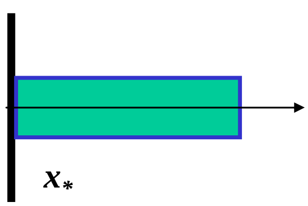
- specified slope, $\left(\partial v \big/ \partial x \right) = \theta_*$

Natural BCs:

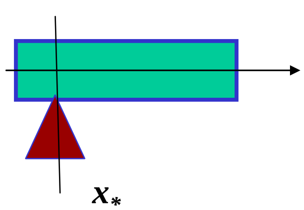
- specified moment, $$M = M_* \;= E\,I \left(\frac{\partial^2 v}{\partial x^2} \right]_{x_*}$$

- specified shear force, $$S = S_* \;= \frac{\partial}{\partial x} \left(E\,I \frac{\partial^2 v}{\partial x^2} \right)_{x_*}$$

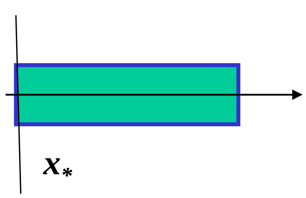
Below are shown the most typical configurations:



Fixed end (cantilever): $v = 0 \;\&\; \dfrac{\partial v}{\partial x} = 0$

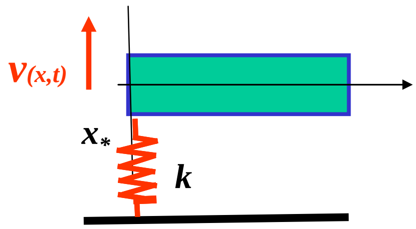


Pinned end $v = 0 \;\& M = 0 \rightarrow \dfrac{\partial^2 v}{\partial x^2} = 0$



Free end

$$M = 0 \;\& S = 0 \rightarrow \frac{\partial^2 v}{\partial x^2} = 0 \;\&\; \frac{\partial^3 v}{\partial x^3} = 0$$



Spring supported end

$$M = \frac{\partial^2 v}{\partial x^2} = 0$$

$$-S = k\, v_* = -\frac{\partial}{\partial x}\left(E\,I \frac{\partial^2 v}{\partial x^2} \right)_{x_*}$$

Note: PDE (44) and its BCs can be derived from the Hamiltonian principle using the definitions for kinetic (*T*) and potential (*V*) energies of an elastic beam

$$T = \frac{1}{2}\int_0^L \rho\, A \left(\frac{\partial v}{\partial t} \right)^2 dx; \quad V = \frac{1}{2}\int_0^L E\,I \left(\frac{\partial^2 v}{\partial x^2} \right)^2 dx \qquad (45)$$

Free vibrations of elastic beam

Without external forces (point loads or distributed load, *f=0*), PDE (44) reduces to

$$\left(\rho\, A \right) \frac{\partial^2 v}{\partial t^2} = -\frac{\partial^2}{\partial x^2}\left(E\,I \frac{\partial^2 v}{\partial x^2} \right) \qquad (46)$$

The solution of PDE (46) is of the form $v_{(x,t)} = \phi_{(x)} \mathrm{V}_{(t)}$ (47)

Let $\left(\dot{\ } \right) = {}^{d}\!/\!_{dt};\ \left({}' \right) = {}^{d}\!/\!_{dx}$. Substituting Eq. (47) into Eq (46) gives

$$\phi_{(x)} \ddot{\mathrm{V}}_{(t)} = \frac{E\,I}{\rho\, A} \mathrm{V} \frac{d^4 \phi_{(x)}}{d\,x^4} \quad \Rightarrow \quad \frac{\ddot{\mathrm{V}}_{(t)}}{\mathrm{V}} = \frac{E\,I}{\rho\, A} \frac{1}{\phi_{(x)}} \frac{d^4 \phi_{(x)}}{d\,x^4} = -\omega^2$$

Above, the *LHS* is only a function of time, while the *RHS* is only a function of spatial coordinate *x*. This is possible only if both are equal to a constant, i.e. $\left(-\omega^2\right)$. Hence, the separation of variables gives two ordinary differential equations

$$\ddot{\mathrm{v}}_{(t)} + \omega^2\, \mathrm{v} = 0 \quad \& \quad \frac{d^4\phi}{d\,x^4} - \lambda^2\,\phi = 0 \tag{48}$$

where

$$\lambda^2 = \omega^2 \left(\frac{\rho\, A}{E\, I}\right) \tag{49}$$

The solution of the ODEs is

$$v_{(t)} = C_t \cos\left(\omega t\right) + S_t \sin\left(\omega t\right) \tag{50}$$

$$\phi_{(x)} = C_1 \cos\left(\beta\, x\right) + C_2 \sin\left(\beta\, x\right) + C_3 \cosh\left(\beta\, x\right) + C_4 \sinh\left(\beta\, x\right)$$

(51)[1]

where

$$\beta = \lambda^{1/2} = \omega^{1/2} \left(\frac{\rho\, A}{E\, I}\right)^{1/4} \tag{52}$$

has units of [1/length].

The coefficients (C,S) are determined from satisfying the boundary conditions for the specific beam configuration. Equation (51) is known as the **fundamental mode shape** for an elastic beam, i.e. it contains the information on natural frequencies and mode shapes.

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[1] The solution of ODE $\phi^{iv} - \lambda^2 \phi = 0 = \phi^{iv} - \beta^4 \phi = 0$ is $\phi = c\, e^{kx}$ with characteristic equation $k^4 - \lambda^2 = 0$

Example 1. Pin-Pin beam

The BCs are:

At x=0, $$v_{(0,t)} = 0 = \phi_{(0)} \mathrm{v}_{(t)} \Rightarrow \phi_{(0)} = 0 \quad \forall t \qquad (53.a)$$

$$\rightarrow \qquad \phi_{(0)} = C_1 + C_3$$

$$M = {\partial^2 v}/{\partial x^2} = 0 = \phi''_{(0)} \mathrm{v}_{(t)} \Rightarrow \phi''_{(0)} = 0$$

$$\rightarrow \qquad \phi''_{(0)} = -C_1 + C_3$$

Hence, C_1=C_3=0 and $\phi_{(x)} = C_2 \sin(\beta x) + C_4 \sinh(\beta x)$

At x=L, $$v_{(L,t)} = 0 = \phi_{(L)} \mathrm{v}_{(t)} \Rightarrow \phi_{(L)} = 0 \quad \forall t$$

$$\rightarrow \qquad \phi_{(L)} = 0 = C_2 \sin(\beta L) + C_4 \sinh(\beta L)$$

$$M_{x=L} = {\partial^2 v}/{\partial x^2} = 0 = \phi''_{(L)} \mathrm{v}_{(t)} \Rightarrow \phi''_{(L)} = 0 \qquad (53.b)$$

$$\rightarrow \qquad \phi''_{(L)} = 0 = -C_2 \sin(\beta L) + C_4 \sinh(\beta L)$$

from this two equations, since $\sinh(\beta L) \neq 0$, it follows that

$$\phi_{(x)} = C_2 \sin(\beta x) \qquad (54)$$

where $\sin(\beta L) = 0$ when $\beta_i = \frac{i\pi}{L}, \quad {}_{i=1,2....\infty}$ (55)

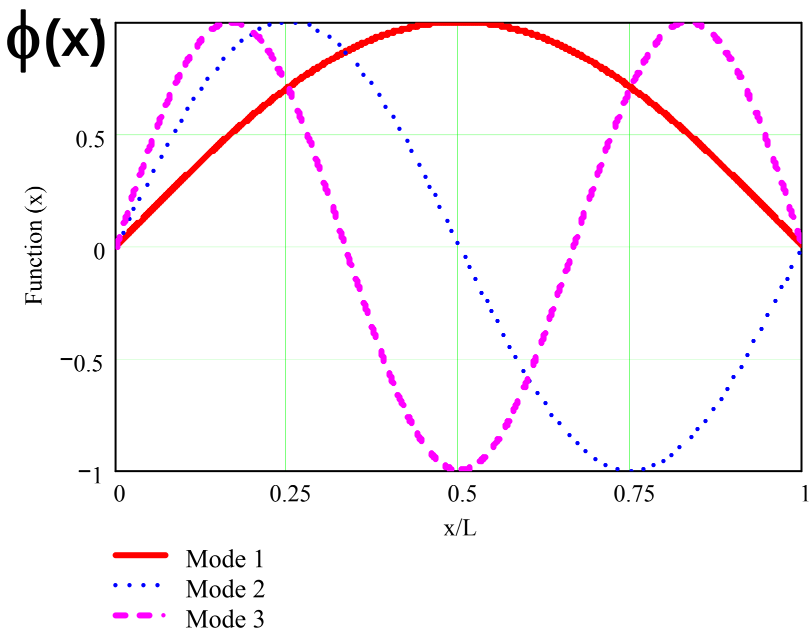
and hence,
the **natural frequencies of the pin-pin beam** are

$$\omega_i = \beta_i^2 \left(\frac{E\,I}{\rho\,A} \right)^{1/2} = \frac{i^2\,\pi^2}{L^2} \left(\frac{E\,I}{\rho\,A} \right)^{1/2} ; \quad {}_{i=1,2...\infty} \tag{56}$$

Associated to each natural frequency, there is a **natural mode shape**

$$\phi_i = \sin\left(\beta_i\, x\right) = \sin\left(\frac{i\,\pi\,x}{L} \right); \quad {}_{i=1,2,.....} \tag{57}$$

as shown in the graph below.



$$\beta = \begin{pmatrix} 3.142 \\ 6.283 \\ 9.425 \end{pmatrix} \frac{1}{L}$$

$$\beta_i := i \cdot \frac{\pi}{L}$$

Fig. Natural mode shapes ϕ(x) for elastic beam with both ends pinned.

The displacement function response $v_{(x,t)} = \phi_{(x)} \mathrm{V}_{(t)}$ equals to the superposition of all the found responses, i.e.

$$v_{(x,t)} = \sum_k \phi(x)_k \, \mathrm{v}(t)_k = \sum_{k=1}^{\infty} \phi_{(x)_k} \left[C_k \cos(\omega_k t) + S_k \sin(\omega_k t) \right]$$

$$v_{(x,t)} = \sum_{k=1}^{\infty} \sin(\beta_k x) \left[C_k \cos(\omega_k t) + S_k \sin(\omega_k t) \right] \quad (58)$$

and velocity:

$$\dot{v}_{(x,t)} = \sum_{k=1}^{\infty} \sin(\beta_k x)\, \omega_k \left[-C_k \sin(\omega_k t) + S_k \cos(\omega_k t) \right] \quad (59)$$

The set of coefficients (C_k, S_k) are determined by satisfying the initial conditions. That is at time $t=0$,

$$\begin{aligned} v_{(x,0)} &= V_{(x)} = \sum_{k=1}^{\infty} \sin(\beta_k x)\, C_k \\ \dot{v}_{(x,0)} &= \dot{V}_{(x)} = \sum_{k=1}^{\infty} \omega_k \sin(\beta_k x)\, S_k \end{aligned} \quad (60)$$

RECALL:

$$\phi_{(x)} = C_1 \cos(\beta x) + C_2 \sin(\beta x) + C_3 \cosh(\beta x) + C_4 \sinh(\beta x)$$

$$\phi' / \beta = -C_1 \sin(\beta x) + C_2 \cos(\beta x) + C_3 \sinh(\beta x) + C_4 \cosh(\beta x)$$

$$\phi'' / \beta^2 = -C_1 \cos(\beta x) - C_2 \sin(\beta x) + C_3 \cosh(\beta x) + C_4 \sinh(\beta x)$$

$$\phi''' / \beta^3 = C_1 \sin(\beta x) - C_2 \cos(\beta x) + C_3 \sinh(\beta x) + C_4 \cosh(\beta x)$$

Example 2. Fixed end-Free end beam

The BCs. are

At $x=0$, $$v_{(0,t)} = 0 = \phi_{(0)} \mathrm{v}_{(t)} \Rightarrow \phi_{(0)} = 0 \quad \forall t \qquad (61.a)$$

$$\rightarrow \quad \phi_{(0)} = C_1 + C_3$$

$$\theta = {}^{\partial v}\!/_{\partial x} = 0 = \phi'_{(0)} \mathrm{v}_{(t)} \Rightarrow \phi'_{(0)} = 0 \qquad (61.b)$$

$$\rightarrow \quad \phi'_{(0)} = C_2 + C_4$$

At $x=L$

$$M_{x=L} = {}^{\partial^2 v}\!/_{\partial x^2} = 0 = \phi''_{(L)} \mathrm{v}_{(t)} \Rightarrow \phi''_{(L)} = 0 \qquad (61.c)$$

$\rightarrow$

$$\phi''_{(L)} = 0 = -C_1 \cos(\beta L) - C_2 \sin(\beta L) + C_3 \cosh(\beta L) + C_4 \sinh(\beta L)$$

$$S_{x=L} = {}^{\partial^3 v}\!/_{\partial x^3} = 0 = \phi'''_{(L)} \mathrm{v}_{(t)} \Rightarrow \phi'''_{(L)} = 0 \qquad (61.d)$$

$\rightarrow$

$$\phi'''_{(L)} = 0 = C_1 \sin(\beta L) - C_2 \cos(\beta L) + C_3 \sinh(\beta L) + C_4 \cosh(\beta L)$$

Solution of Eqs. (a)-(d) gives

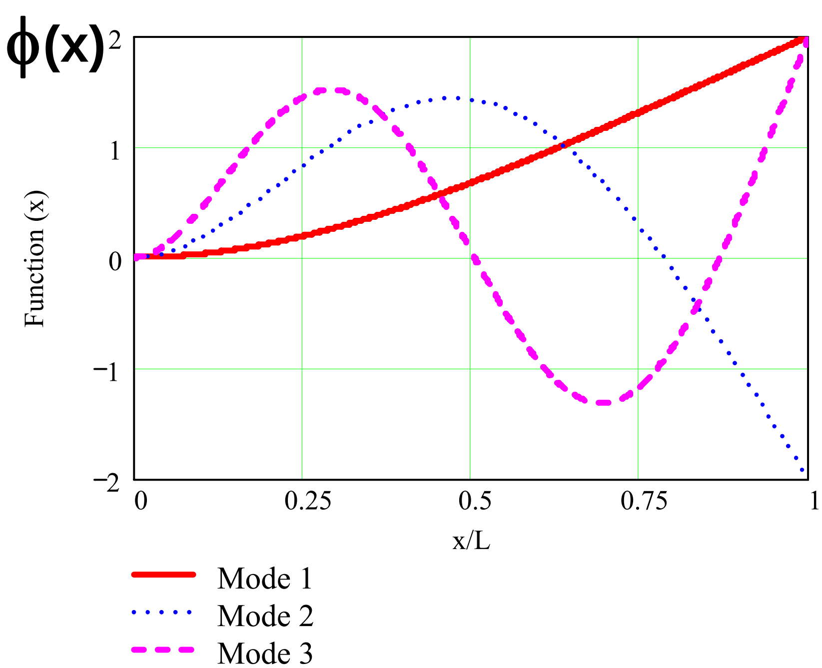
$$\phi_{(x)} = \cosh(\beta_i x) - \cos(\beta_i x) - \alpha_i \left[\sinh(\beta_i x) - \sin(\beta_i x)\right] \quad :$$
(62)

where
$$\alpha_i = \frac{\cosh(\beta_i L) + \cos(\beta_i L)}{\sinh(\beta_i L) + \sin(\beta_i L)} \tag{63}$$

and

$$\begin{aligned} \beta_1 L &= 1.875104 \rightarrow \alpha_1 = 0.734096 \\ \beta_2 L &= 4.694041 \rightarrow \alpha_2 = 1.018466 \\ \beta_3 L &= 7.854757 \rightarrow \alpha_3 = 0.999225 \\ &etc \end{aligned} \tag{64}$$

$$\phi(\beta, x) := \cosh(\beta \cdot x) - \cos(\beta \cdot x) - \alpha(\beta) \cdot (\sinh(\beta \cdot x) - \sin(\beta \cdot x))$$



$$\beta = \begin{pmatrix} 1.875 \\ 4.694 \\ 7.855 \end{pmatrix} \frac{1}{L}$$

$$\alpha\_ = \begin{pmatrix} 0.734 \\ 1.018 \\ 0.999 \end{pmatrix}$$

Fig. Natural mode shapes $\phi(x)$ for cantilever beam (fixed-free ends)

Properties of the natural modes

Recall that the pair $\left\{\lambda_k, \phi_{(x)_k}\right\}_{k=1,\ldots\infty}$ satisfy the ODE

$$\phi^{iv}{}_k - \lambda_k^2\, \phi_k = 0 \qquad k=1,2\ldots\infty \tag{65}$$

where $\qquad \beta_k^4 = \lambda_k^2 = \omega_k^2 \left(\dfrac{\rho A}{E I}\right)$

As in the case of axial vibrations of a bar, it is easy[2] to show that the natural modes $\left\{\phi_k\right\}_{k=1,2\ldots}$ of a flexing beam satisfy the following **ORTHOGONAL** properties:

$$\int_0^L \left(E A \phi_i'' \phi_j''\right) dx = \begin{cases} \mathrm{K}_i & \text{for } i=j \\ 0 & \text{for } i \neq j \end{cases} \tag{66a}$$

$$\int_0^L \left(\rho A \phi_i \phi_j\right) dx = \begin{cases} \mathrm{M}_i & \text{for } i=j \\ 0 & \text{for } i \neq j \end{cases} \tag{66b}$$

For $i=j$, the i_{th} natural frequency follows from

$$\omega_i^2 = \frac{\mathrm{K}_i}{\mathrm{M}_i} = \frac{\int_0^L \left(E A \left(\phi_i''\right)^2\right) dx}{\int_0^L \left(\rho A \phi_i^2\right) dx} \tag{67}$$

Where $\mathrm{K}_i, \mathrm{M}_i$ are the i_{th} mode *equivalent* stiffness and mass coefficients.

[2] Demonstration with integration by parts (twice).

Note that $\{\phi_k\}_{k=1,2...}$ is a **COMPLETE SET** of orthogonal functions

Now, consider the **initial conditions** for

$$v_{(x,t)} = \sum_k \phi(x)_k \, \mathrm{v}(t)_k = \sum_{k=1}^{\infty} \phi_{(x)_k} \left[C_k \cos(\omega_k t) + S_k \sin(\omega_k t) \right]$$

$$v_{(x,0)} = V_{(x)} = \sum_{k=1}^{\infty} \phi_k \, C_k ; \quad \dot{v}_{(x,0)} = \dot{V}_{(x)} = \sum_{k=1}^{\infty} \phi_k \omega_k \, S_k \qquad (68)$$

Using the orthogonality properties, the coefficients (C_m, S_m) follow from

$$C_m = \frac{\int_0^L \left(\rho A \phi_m V_{(x)} \right) dx}{\mathrm{M}_m}, \quad m=1,2,...\infty \qquad (69a)$$

And similarly

$$S_m = \frac{\int_0^L \left(\rho A \phi_m \dot{V}_{(x)} \right) dx}{\omega_m \mathrm{M}_m}, \quad m=1,2,...\infty \qquad (69b)$$

This concludes the procedure to obtain the full solution for the lateral vibrations of a beam, i.e.

$$v_{(x,t)} = \sum_{k=1}^{\infty} \phi_{(x)_k} \left[C_k \cos(\omega_k t) + S_k \sin(\omega_k t) \right] \qquad (70)$$

Forced lateral vibrations of a beam

Consider a beam subjected to an arbitrary forcing function $f_{(x,t)}$. The PDE describing the lateral motions of the beam is

$$(\rho A)\frac{\partial^2 v}{\partial t^2} = f_{(x,t)} - \frac{\partial^2}{\partial x^2}\left(E\,I\,\frac{\partial^2 v}{\partial x^2}\right) \tag{44}$$

Let $\{\phi_k\}_{k=1,2...}$ be the set of natural modes satisfying the boundary conditions of the beam configuration (pin-pin, fixed-free ends, etc). A solution to Eq. (44) is of the form

$$v_{(x,t)} = \sum_{k=1}^{\infty} \phi_{(x)_k}\, q_{(t)_k} \tag{71}$$

Since the set $\{\phi_k\}_{k=1,2...}$ is complete, then any arbitrary function $f_{(x,t)}$ can be written as

$$f_{(x,t)} = \sum_{k=1}^{\infty} \phi_{(x)_k}\, Q_{(t)_k} \tag{72}$$

where

$$Q_m = \frac{\int_0^L \left(\rho\, A\, \phi_m\, f_{(x,t)}\right) dx}{\mathrm{M}_m}, \quad m=1,2,...\infty \tag{73}$$

Substitution of Eqs. (71, 72) into Eq. (44) $(\rho A)\frac{\partial^2 v}{\partial t^2} = f_{(x,t)} - \frac{\partial^2}{\partial x^2}\left(E\,I\,\frac{\partial^2 v}{\partial x^2}\right)$

gives

$$\sum_{k=1}^{\infty}\left[\rho A \phi_k \ddot{q}_k - \phi_k Q_k + E I \phi_k^{iv} q_k\right] = 0 \qquad (74)$$

but recall that each of the normal modes satisfies $\phi_k^{iv} - \lambda_k^2 \phi_k = 0$; and hence, Eq. (74) can be written as

$$\sum_{k=1}^{\infty}\left[\rho A \ddot{q}_k - Q_k + E I \lambda_k^2 q_k\right]\phi_k = 0$$

and, since the natural modes are linearly independent, then it follows that

$$\rho A \ddot{q}_k - Q_k + E I \lambda_k^2 q_k = 0 \quad {}_{k=1,2,\ldots\infty} \qquad (75)$$

Lastly, recall that $\lambda_k^2 = \omega^2\left(\dfrac{\rho A}{E I}\right)$; then $\lambda_k^2 E I = \omega^2 \rho A$, and write (75) as

$$\ddot{q}_k + \omega_k^2 q_k = {Q_k}\big/{\rho A} \quad ; \quad {}_{k=1,2,\ldots\infty} \qquad (76)$$

Which can be easily solved for all type of excitations $Q_{(t)_k}$

[See solution of undamped SDOF EOMS – Lectures #2]

Example 3. Free Ends beam

The BCs are:

At $x=0$

$$M = {}^{\partial^2 v}\!/\!_{\partial x^2} = 0 = \phi''_{(0)} \mathrm{v}_{(t)} \Rightarrow \quad \phi''_{(0)} = 0$$

$\rightarrow$

$$\phi''_{(0)} = -C_1 + C_3 \quad \text{(a)}$$

$$S_{x=0} = {}^{\partial^3 v}\!/\!_{\partial x^3} = 0 = \phi'''_{(0)} \mathrm{v}_{(t)} \Rightarrow \quad \phi'''_{(0)} = 0$$

$\rightarrow$

$$\phi'''_{(0)} = -C_2 + C_4 \quad \text{(b)}$$

At $x=L$

$$M_{x=L} = {}^{\partial^2 v}\!/\!_{\partial x^2} = 0 = \phi''_{(L)} \mathrm{v}_{(t)} \Rightarrow \quad \phi''_{(L)} = 0 \qquad (61.c)$$

$\rightarrow$

$$\phi''_{(L)} = 0 = -C_1 \cos(\beta L) - C_2 \sin(\beta L) + C_3 \cosh(\beta L) + C_4 \sinh(\beta L)$$

(c)

$$S_{x=L} = {}^{\partial^3 v}\!/\!_{\partial x^3} = 0 = \phi'''_{(L)} \mathrm{v}_{(t)} \Rightarrow \quad \phi'''_{(L)} = 0$$

$\rightarrow$

$$\phi'''_{(L)} = 0 = C_1 \sin(\beta L) - C_2 \cos(\beta L) + C_3 \sinh(\beta L) + C_4 \cosh(\beta L)$$

(d)

Solution of Eqs. (a)-(d) gives

$$\phi_{(x)} = \cosh(\beta_i x) - + \cos(\beta_i x) - \alpha_i \left[\sinh(\beta_i x) + \sin(\beta_i x) \right]$$

where

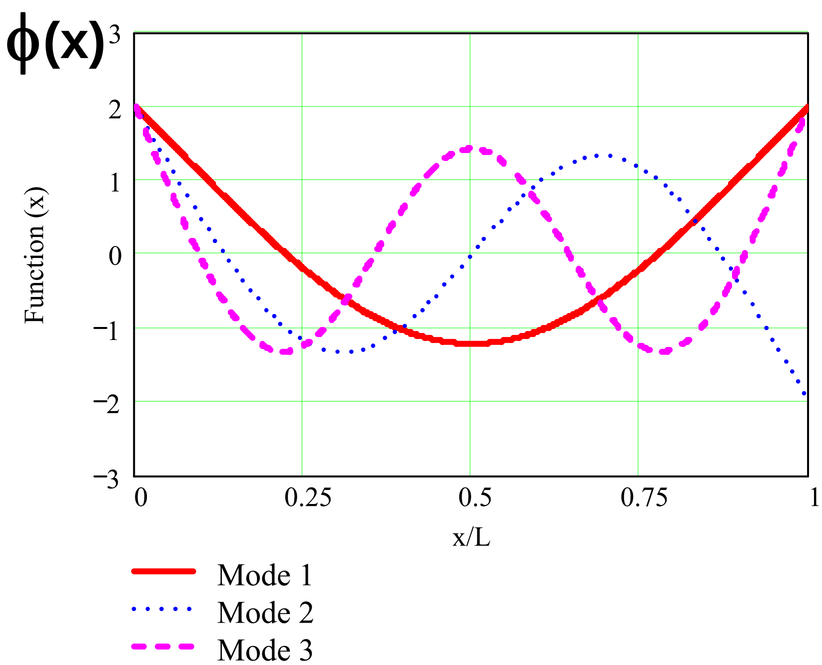
$$\alpha_i = \frac{\cosh(\beta_i L) - \cos(\beta_i L)}{\sinh(\beta_i L) - \sin(\beta_i L)} \qquad (63)$$

and

$$\begin{aligned}
&\beta_1 L = 4.730041 \rightarrow \alpha_1 = 0.982502\\
&\beta_2 L = 7.853205 \rightarrow \alpha_2 = 1.000777\\
&\beta_3 L = 10.99560 \rightarrow \alpha_3 = 0.999966\\
&etc
\end{aligned} \tag{64}$$

Note that the lowest natural frequency is actually zero, i.e. a rigid body mode. β_0=0 & $\phi_0 = 1$

$\phi(\beta, x) := \cosh(\beta \cdot x) + \cos(\beta \cdot x) - \alpha(\beta)\cdot(\sinh(\beta \cdot x) + \sin(\beta \cdot x))$



$$\beta = \begin{pmatrix} 4.73 \\ 7.853 \\ 10.996 \end{pmatrix} \frac{1}{L}$$

$$\alpha\_ = \begin{pmatrix} 0.983 \\ 1.001 \\ 1 \end{pmatrix}$$

Fig. Elastic natural mode shapes ϕ(x) for beam with free ends)

Characteristic (mode shape) equation for beams:

$$\phi_{(x)} = C_1 \cos(\beta x) + C_2 \sin(\beta x) + C_3 \cosh(\beta x) + C_4 \sinh(\beta x)$$

$$\phi' / \beta = -C_1 \sin(\beta x) + C_2 \cos(\beta x) + C_3 \sinh(\beta x) + C_4 \cosh(\beta x)$$

$$\phi'' / \beta^2 = -C_1 \cos(\beta x) - C_2 \sin(\beta x) + C_3 \cosh(\beta x) + C_4 \sinh(\beta x)$$

$$\phi''' / \beta^3 = C_1 \sin(\beta x) - C_2 \cos(\beta x) + C_3 \sinh(\beta x) + C_4 \cosh(\beta x)$$